where Ω is the number of states ν in region II. With this we obtain the criterion

$$\sqrt{\Omega} \gg 1. \tag{C7}$$

The larger Ω is, the more reliable the saddle-point method should be. Large Ω is favored by strong pairing interactions. On the other hand, the BCS state conserves particle number in the case of a sharp Fermi surface. Thus, it is to be expected that in the limit of very small Ω the errors connected with the nonconservation of particle number should be small too,[9] whereas the saddle-point method can be substantially wrong in this case. This is borne out by our numerical results (see Sec. V).

PHYSICAL REVIEW VOLUME 135, NUMBER 1B 13 JULY 1964

Separability of Center-of-Mass Motion in the Nuclear Shell Model

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It is shown that one can find an orthogonal transformation that will enable one to split the motion of a many-body system to a center-of-mass motion and an internal motion. A particular orthogonal transformation has been chosen which retains the independent-particle aspect of a harmonic-oscillator shell-model Hamiltonian. It is suggested that one could easily study and eliminate the states with spurious motions of the center of mass by a direct transformation of the shell-model wave function into the new coordinate system.

1. INTRODUCTION

IN calculations of nuclear structure using shell-model wave functions, the shell-model wave functions are constructed from independent-particle wave functions. These independent-particle wave functions describe the motion of a particle moving in a potential fixed in space. Because of the assumption of a potential fixed in space, the shell-model wave functions so obtained are not translationally invariant. It has been recognized that the neglect of the center-of-mass motion will cause errors in the calculations of energies and transition matrix elements. But the extraction of the center-of-mass motion leaves us only (A−1) degrees of freedom and the internal coordinates cannot be treated symmetrically; therefore, the construction of antisymmetric states becomes very cumbersome.

It was first shown by Bethe and Rose[1] that the antisymmetrized shell-model wave function for the lowest states in a harmonic-oscillator potential are always translationally invariant. However, some of the excited states of the nucleus could be describing a system whose center of mass is in motion. These "spurious states" were first recognized by Elliott and Skyrme.[2] They pointed out that when two or more unclosed shells are involved, one has to investigate that the state has the proper center-of-mass motion. Their prescription is to form suitable linear combinations of shell-model wave functions to describe the proper center-of-mass motion.

In our investigations, we have tried to find a coordinate transformation such that the total kinetic-energy operator of the many-body Hamiltonian splits up into the center-of-mass kinetic energy and the remaining kinetic energy of relative motion. This has been done for the case of two degrees of freedom by Talmi[3] and Theiberger.[4] In the particular case of the harmonic-oscillator potential, our choice of orthogonal transformation separates the center-of-mass part of the potential energy for a general n-body problem.

2. THE COORDINATE TRANSFORMATION

We shall designate the original A-independent set of coordinates by $(\gamma_1,\gamma_2\cdots,\gamma_A)$, and the transformed coordinates by $(\xi_0,\xi_1\cdots\xi_{A-1})$. The transformation[5] is given by[6]

$$\xi_i=\gamma_i-\frac{1}{A-i}\sum_{j=i+1}^{A}\gamma_j \quad i=0,1\cdots A-1$$
$$\gamma_0=0. \tag{2.1}$$

The inverse transformation is given by

$$\gamma_i=\xi_i-\xi_0-\sum_{j=1}^{i}\frac{1}{(A-j+1)}\xi_j \quad i=1\cdots A$$
$$\xi_A=0. \tag{2.2}$$

[1] H. A. Bethe and M. E. Rose, Phys. Rev. **51**, 283 (1937).
[2] J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A232**, 561 (1955).
[3] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[4] R. Thieberger, Nucl. Phys. **2**, 533 (1956).
[5] Note that our first coordinate is $\xi_0=-\mathbf{R}$, and is thus different from the corresponding one of Lipperheide. Our choice makes the transformation orthogonal.
[6] R. Lipperheide, Ann. Phys. (N. Y.) **17**, 114 (1962); S. Hochberg, H. S. W. Massey, and L. H. Underhill, Proc. Phys. Soc. (London) **A67**, 957 (1959).

It can be verified that with a suitable normalization, the transformation matrix can be made orthogonal.

Substituting Eqs. (2.1), we obtain the following expression for the kinetic energy operator

$$-\frac{\hbar^2}{2m}\sum_{i=1}^{A}\nabla_{\gamma_i}^2=-\frac{\hbar^2}{2Am}\nabla_{\xi_0}^2-\frac{\hbar^2}{2m}\sum_{j=1}^{A-1}\left(\frac{A-j+1}{A-j}\right)\nabla_{\xi_j}^2. \tag{2.3}$$

The first term on the right in Eq. (2.3) represents the kinetic energy of the center-of-mass coordinate. If $\mathbf{R}=A^{-1}\sum_{i=1}^{A}\boldsymbol{\gamma}_i$ is the center-of-mass coordinate, $\xi_0=-R$. Because the transformation is orthogonal, we have no cross terms in the kinetic energy. For the particular case when the potential-energy operator of the model Hamiltonian is of the form

$$V=k\sum_{i=1}^{A}\gamma_i^2, \tag{2.4}$$

the transformed potential energy has the form

$$V=kA\xi_0^2+k\sum_{j=1}^{A-1}\left(\frac{A-j}{A-j+1}\right)\xi_j^2, \tag{2.5}$$

i.e., the potential energy again resolves as a sum of single-particle harmonic-oscillator potentials. The total model Hamiltonian has the form

$$\begin{aligned}H_0&=-\frac{\hbar^2}{2m}\sum_{i=1}^{A}\nabla_{\gamma_i}^2+k\sum_{i=1}^{A}\gamma_i^2\\&\equiv\left(-\frac{\hbar^2}{2Am}\nabla_{\xi_0}^2+kA\xi_0^2\right)+\sum_{j=1}^{A-1}\left(-\frac{\hbar^2}{2m_j}\nabla_{\xi_j}^2+k_j\xi_j^2\right),\end{aligned} \tag{2.6}$$

where $m_j=(A-j)m/(A-j+1)$ is the reduced mass of the jth nucleon and $k_j=(A-j)k/(A-j+1)$ is the oscillator constant for the jth nucleon. (Classically, one could say that the problem of A equally massive simple harmonic oscillators with equal oscillator strength has been transformed to a problem of A simple harmonic oscillators of different masses and different oscillator strengths, but each still has the same frequency of vibration.)

Equation (2.6) describes a system whose center of mass is moving in a harmonic-oscillator potential, and whose internal motions are superpositions of simple harmonic motions. The fact that the center of mass still has a zero-point energy is a feature of the harmonic oscillator Hamiltonian. A requirement of translational invariance in this case would be to force the center of mass to be in its lowest S state of motion. The fact that the resulting internal Hamiltonian is a sum of independent particle Hamiltonian seems to be because of our particular choice of potential energy. In general, one does not expect the potential energy to transform in such a simple fashion.

The eigenfunctions of the internal Hamiltonian will have their radial functions of the form

$$R(\xi)=P(\xi)\exp(-\tfrac{1}{2}\textstyle\sum_j \nu_j\xi_j^2), \tag{2.7}$$

where $\nu_j^2=m_jk_j/2\hbar^2$ and $P(\xi)$ is a product of Hermite polynomials.

The total energy of a proper shell-model state would be given by

$$E=E_{\text{int}}+E_R, \tag{2.8}$$

where $E_R=E_{\xi_0}$ is the center-of-mass energy and is equal to $\frac{3}{2}\hbar\omega$ where $\hbar\omega$ is the oscillator quantum, and E_{int} is the eigenvalue for internal motion associated with the $R(\xi)$ given by Eq. (2.7).

3. CONCLUSION

We have seen that one could choose a coordinate transformation that would retain the single-particle nature of the problem and at the same time separate the center-of-mass motion. The separation of the center-of-mass part of the potential is a particular feature of the harmonic-oscillator nature of the potential. We had chosen the simple case of a harmonic-oscillator potential because these are the most used wave functions in the calculations of nuclear structure. The new internal coordinates are not symmetric and hence would make the problem of antisymmetrization difficult. However, each of the new coordinates chosen has a geometrical significance. The first coordinate ξ_0 is related to the center of mass of the whole system, the second coordinate represents the separation of particle one from the center of mass of the remaining system, and so on. Hence, the above transformation may be useful in estimating the recoil corrections in nuclear reactions.

The residual two-body force which causes an energy splitting in the nuclear-shell-model calculation will have the form of a many-body force in the center-of-mass coordinate system. However, if we use completely antisymmetric wave functions, the many-body nature of the perturbing potential will not cause any difficulty, because the matrices of each of the two-body interactions will yield identical results. In the orthogonal transformation chosen, one of the coordinates, $\xi_{A-1}=\gamma_{A-1}-\gamma_A$, is just the difference of two radius vectors, and we could choose to calculate the matrices of $V(\xi_{A-1})$.[7]

Because of the difficulty involved in the antisymmetrization of the product wave function, one is usually tempted to define all the coordinates with respect to the center of mass. However, the resulting transformation is not orthogonal and we do not obtain a convenient grouping of kinetic-energy terms.

In order to retain the advantages of calculations with

[7] B. F. Bayman (private communication).

the product wave functions without substracting the center-of-mass coordinate, we could make the following prescription to eliminate spurious states.[8] A state with the proper center-of-mass motion can be written as

$$\Psi(\gamma_1,\gamma_2\cdots\gamma_A)=e^{-\nu A\xi_0^2}\Phi_{\rm int}(\xi_1\cdots\xi_{A-1}).$$

Before we proceed to do our calculations with our original wave function, we could transform it into the new system and verify whether the center-of-mass wave function has the desired form. The above prescription can be shown to be identical to that of Elliott and Skyrme.[2]

If we had started with a potential energy in the more natural form $k\sum_{i=1}^{A}(\gamma_i-\mathbf{R})^2$, where R is the center of mass, the center-of-mass Hamiltonian would consist only of the kinetic-energy term and all the other features would still remain the same.

In a more realistic calculation, if the average potential is quite close to a harmonic-oscillator potential, one could make a perturbation approach, studying the variation of energy due to a perturbation of the harmonic-oscillator potential. This method was suggested to the author by Professor R. M. Thaler. Some calculations using this approach are being done presently.

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Decay of Rb^{90}

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Following the neutron irradiation of uranyl stearate, Rb^{90} sources were prepared by milking from the emanated Kr^{90}. The decay properties of Rb^{90} were then investigated with scintillation techniques. From single-crystal and gamma-gamma coincidence spectra, gamma rays (and intensities) were found at energies of 0.53(6.6), 0.72(6.5), 0.832(90), 0.86(10), 1.03(6.0), 1.11(13), 1.24(4.3), 1.40(8.3), 1.70(6.1), 1.82(4.8), 2.20(3.6), 2.51(5.6), 2.72(1.8), 2.84(3.0), 3.07(9.7), 3.34(28), 3.54(12), 4.13(20), 4.34(24), 4.37(7.8), 4.60(8.4), 5.08(3.0), and 5.23(6.7) MeV. Also, from single-crystal and beta-gamma coincidence spectra, beta-ray groups were observed with energies of 1.31, 2.0, 2.2 (complex), 4.37, 5.81, and 6.60 MeV, where the last group represents the separation between the Rb^{90}-Sr^{90} ground states. The decay scheme proposed for Rb^{90} involves levels in Sr^{90} at 0.832, 1.69, 1.94, 2.23, 3.07, 3.34, 4.15, 4.34, 4.60, 5.08, and 5.23 MeV. The half-lives of Rb^{90} and Rb^{91} have been remeasured as 2.91±0.05 min and 1.17±0.10 min, respectively.

I. INTRODUCTION

As part of a program of surveying the properties of several nuclear families of constant Z, we have previously studied[1,2] the levels of Sr^{88} and Sr^{89}. This effort has now been extended to the nucleus Sr^{90} by observing the decay radiations from Rb^{90}.

The nucleus Rb^{90} was first characterized by Kofoed-Hansen and Nielsen.[3] They swept the rare gases from a neutron-irradiated uranium solution into a magnetic isotope separator. By separating Kr^{90} and allowing it to decay, they produced an essentially pure Rb^{90} sample. They reported a 2.74-min half-life and a maximum beta-ray end-point energy of 5.7 MeV as determined by aluminum absorption. Wahlgren[4] also observed Rb^{90} in his study of Kr^{90}.

We have utilized beta- and gamma-ray scintillation techniques in our investigation of the Rb^{90} decay. A preliminary report of the present work has appeared earlier.[5]

II. SOURCE PREPARATION AND HALF-LIFE DETERMINATION

The Rb^{90} sources were obtained by decay of the rare-gas parent, 33-sec Kr^{90}, which was emanated from a neutron-irradiated uranium sample. The experimental

* Operated by the Union Carbide Corporation, Nuclear Division, for the U. S. Atomic Energy Commission.

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